

Introduction to chromatic homotopy

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July 18, 2019

Ordinary cohomology

- ▶ For any space X we have a cohomology ring $H^*(X)$
- ▶ For many spaces this can be described explicitly: for example, if $X = \{ \text{two-dimensional subspaces of } \mathbb{C}^4 \}$ then $H^*(X) = \mathbb{Z}[c_1, c_2]/(c_1^3 - 2c_1c_2, c_1^2c_2 - c_2^2)$.
- ▶ We can also consider the scheme $X_H = \text{spec}(H^*(X))$, so $H^*(X)$ is the ring of functions on X_H .
- ▶ Now $f: X \rightarrow Y$ gives $f_H: X_H \rightarrow Y_H$ (depending only on the homotopy class) and $(X \amalg Y)_H = X_H \amalg Y_H$ and $(X \times Y)_H \sim X_H \times Y_H$.
- ▶ How good an invariant is this?
 - ▶ If $f_H: X_H \rightarrow Y_H$ is an isomorphism then f is a homotopy equivalence (subject to mild conditions).
 - ▶ The map $[X, Y] \rightarrow \text{Schemes}(X_H, Y_H) = \text{Rings}(H^*(Y), H^*(X))$ is typically far from being injective or surjective.
 - ▶ If $X_H \simeq Y_H$, that is only weak evidence for $X \simeq Y$.
- ▶ How to find better invariants?
 - (a) Use Steenrod operations on $H^*(X; \mathbb{F}_p)$
 - (b) Use generalised cohomology theories.
- ▶ But (a) is really part of (b).

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- ▶ We often work with *even periodic theories* where $E^1(1) = 0$ and $E^{-2}(1)$ contains a unit. Here it is natural to focus on $E^0(X)$.
- ▶ Given an even periodic theory E we put $X_E = \text{spf}(E^0 X)$.
- ▶ There is an even periodic theory KU with $KU^*(1) = \mathbb{Z}[u, u^{-1}]$ (where $|u| = -2$) and $KU^0(X)$ is the ring of virtual complex vector bundles on X .
- ▶ Put $MP(n) = \{(v, V) \mid v \in V \leq \mathbb{C}^{2n}\}_\infty$ and $\Sigma^m X = (\mathbb{R}^m \times X)_\infty$ and $MP^k(X) = \lim_{\rightarrow n} [\Sigma^{2n-k} X, MP(n)]$.

This gives an even periodic theory with $MP^*(1) = \mathbb{Z}[a_1, a_2, a_3, \dots]$.

This is called *periodic complex cobordism*.

- ▶ The Nilpotence (pre)Theorem of Hopkins-Devnatz-Smith: if $MP^*(u) = 0$ then $u^k = 0$ for large k . This is the most powerful known theorem of the type algebra \Rightarrow topology.
- ▶ Fix a prime p and an integer $n > 0$. There is then an even periodic theory $K(p, n)$ with $K(p, n)^*(1) = \mathbb{F}_p[u, u^{-1}]$. This is called *Morava K-theory*.
- ▶ The $K(p, n)$'s together carry roughly the same information as MP .

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This is called *periodic complex cobordism*.

- ▶ The Nilpotence (pre)Theorem of Hopkins-Devnatz-Smith: if $MP^*(u) = 0$ then $u^k = 0$ for large k . This is the most powerful known theorem of the type algebra \Rightarrow topology.
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- ▶ A *generalised cohomology theory* is a contravariant, homotopy invariant functor $E^* : \text{Spaces} \rightarrow \text{Rings}^*$ with properties similar to H^* , but $E^*(1)$ need not be \mathbb{Z} . It takes work to provide interesting examples.
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Formal groups — what are they good for?

- ▶ Every even periodic theory E gives a formal group P_E .
- ▶ The functor $E \mapsto P_E$ is not too far from being an equivalence.
- ▶ The most elementary examples of formal groups are the additive and multiplicative formal groups; these correspond to HP and KU .
(Here $HP^i(X) = \prod_j H^{i+2j}(X)$.)
- ▶ Steenrod operations in $HP^0(X; \mathbb{F}_p)$ and Adams operations in $KU^0(X)$ are closely related to endomorphisms of the associated formal groups.
- ▶ The ring $MP^0(1)$ is naturally isomorphic to the Lazard ring, which plays a central role in formal group theory.
- ▶ The Morava K -theories $K(p, n)$ all have different formal groups.
- ▶ Together with $HP^0(X; \mathbb{F}_p)$ and $HP^0(X; \mathbb{Q})$ this gives all formal groups over fields up to Galois twisting.
- ▶ For many spaces X the scheme X_E can be described naturally in terms of P_E . For example, if $X = BU(n) = \{n\text{-dimensional subspaces of } \mathbb{C}^\infty\}$ then $X_E = (P_E^n)/\Sigma_n$.

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Examples of formal groups

- ▶ For any ring R we define commutative groups as follows:
 - ▶ $G_a(R) = \{a \in R \mid a \text{ is nilpotent} \}$ (under addition)
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- ▶ These are all functorial in R .
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- ▶ The functors G_i are *formal groups*; the power series F_i are *formal group laws*.
- ▶ Axioms: $F(s, 0) = s$, $F(s, t) = F(t, s)$ and $F(F(s, t), u) = F(s, F(t, u))$.
- ▶ More general version: we have a ground ring k , and $G(R)$ is only functorial for k -algebras, and $F(s, t) \in k[[s, t]]$.
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- ▶ We can calculate $E^*(\mathbb{C}P^n)$ by induction on n using Mayer-Vietoris. It follows that there exists x with $E^0(P) = E^0(1)[[x]]$ (but there is no canonical choice of x).
- ▶ This gives $E^0(P \times P) = E^0(1)[[x_1, x_2]]$. The multiplication map $\mu: P \times P \rightarrow P$ has $\mu^*(x) = F(x_1, x_2)$ for some formal group law F .
- ▶ Now fix a prime p and let $\pi: P \rightarrow P$ be the p 'th power map and put $B = (\mathbb{C}[t] \setminus \{0\})/C_p$.
- ▶ Suppose that $p = 0$ in $E^0(1)$. Under some conditions that are often satisfied, we have $E^0(B) = E^0(1)[[x]]/\pi^*(x)$ and this is free of finite rank over $E^0(1)$. If so, then the rank is always p^n for some $n > 0$, called the *height*.
- ▶ For $E = K(p, n)$ we have $\pi^*(x) = x^{p^n}$ and the height is n .
- ▶ Over an algebraically closed field of characteristic p , any two formal groups of the same height are isomorphic.

Formal groups from even periodic theories

- ▶ $P = (\mathbb{C}[t] \setminus \{0\})/\mathbb{C}^\times = \{1 - \dim \text{ subspaces of } \mathbb{C}[t]\} = \mathbb{C}P^\infty$.
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- ▶ We can calculate $E^*(\mathbb{C}P^n)$ by induction on n using Mayer-Vietoris. It follows that there exists x with $E^0(P) = E^0(1)\llbracket x \rrbracket$ (but there is no canonical choice of x).
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The Lazard ring

- ▶ Consider a formal power series $F(s, t) = \sum_{i,j} b_{ij} s^i t^j \in k[[s, t]]$.
When is this an FGL?
- ▶ For $F(s, 0) = s$ we need $b_{i0} = \delta_{i,1}$. For $F(s, t) = F(t, s)$ we need $b_{ij} = b_{ji}$.
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$$F(s, t) = s + t + b_{11}st + b_{12}(st^2 + s^2t) + b_{22}s^2t^2 + b_{13}(st^3 + s^3t) + O(5)$$
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- ▶ There are no more relations: any power series of the above form satisfies the FGL conditions up to errors of order 5.
- ▶ Lazard's theorem: we can continue to define a_4, a_5, \dots so that $F(s, t)$ can be expressed in terms of the a_i , and no further relations are required to make the associativity axiom hold.
- ▶ Reformulation: over the Lazard ring $L = \mathbb{Z}[a_1, a_2, \dots]$ there is a universal formal group law F_u such that the resulting map $\text{Rings}(L, k) \rightarrow \text{FGL}(k)$ is bijective for all k .

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Quillen's theorem

- ▶ Recall $MP^0(X) = \lim_{\rightarrow n} [\Sigma^{2n} X, MP(n)]$ (for X a finite complex). Both P and $MP(n)$ are defined using complex linear algebra so it is not hard to give an explicit x with $MP^0(P) = MP^0(1)[[x]]$. (We do not need to know $MP^0(1)$ for this.)
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- ▶ Quillen's theorem: this is an isomorphism (and $MP^1(1) = 0$).
- ▶ Outline of proof:
 - ▶ Assemble the spaces $MP(n)$ into a single "spectrum" called MP . (This is the start of stable homotopy theory.)
 - ▶ There are good methods for calculating the homology of spaces defined using complex linear algebra, and one can use them to prove that

$$H_*(MP) = \mathbb{Z}[b_0, b_1, b_2, \dots][[b_0^{-1}]].$$

- ▶ A simple topological construction gives a map $MP^0(1) \rightarrow H_*(MP)$. We can push forward the FGL over $MP^0(1)$ to get an FGL over $H_*(MP)$.
- ▶ In fact this is $F(s, t) = f^{-1}(f(s) + f(t))$, where $f(t) = \sum_i b_i t^{i+1}$. So f gives an isomorphism from F to the additive law $F_a(s, t) = s + t$.
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- ▶ A *finite spectrum* is an expression $\Sigma^n X$, where X is a based finite simplicial complex, and $n \in \mathbb{Z}$. (This can be interpreted as a space if $n \geq 0$, but not necessarily if $n < 0$.) We write \mathcal{F} for the class of finite spectra.
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- ▶ Chromatic convergence theorem: $\pi_*^S(X) = \mathcal{F}(S^*, X)$ can be built up in layers. The difference between layers n and $n - 1$ is in some sense controlled by $K(p, n)$, and consists of families that are periodic of period $2(p^n - 1)p^k$ for large k .

The chromatic filtration

- ▶ Fact: if $K(p, n)_*(X) = 0$, then $K(p, m)_*(X) = 0$ for all $m < n$ (including $K(p, 0)_*(X) = H_*(X; \mathbb{Q})$).
- ▶ Also, if $K(p, n)_*(X) = 0$ for all p and n then $X = 0$.
- ▶ Say X has *type* n at p if $K(p, n)_*(X) \neq 0$ and $K(p, m)_*(X) = 0$ for $m < n$. Let $\mathcal{F}(p, n)$ be the category of X of type at least n at p .
- ▶ Nilpotence theorem: if $u: \Sigma^d X \rightarrow X$ and $K(p, n)_*(u) = 0$ for all (p, n) then $u^k = 0: \Sigma^{dk} X \rightarrow X$ for $k \gg 0$.
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