THE TELESCOPE CONJECTURE

In summer 2023, Burklund, Hahn, Levy and Schlank announced a disproof of Ravenel's Telescope Conjecture, which has been open since 1984. That conjecture stated that two subcategories of spectra were equal; it now seems likely that there is a rich structure of intermediate categories, and we can anticipate an explosion of work to investigate them.

This document is an attempt to list the background needed to understand some of this new progress.

1. Chromatic homotopy theory

- Basics of spectra.
- Complex cobordism, Quillen's theorem on the relationship with formal group laws.
- Classification of FGLs by height.
- Morava K-theory and E-theory, and $BP\langle n\rangle$.
- The nilpotence theorem, classification of thick subcategories of finite spectra, v_n-self maps and telescopes.
- Bousfield localisation with respect to $E(n)$, $K(n)$, $E^f(n)$ and $K^f(n)$. Statement of the Telescope Conjecture.
- A bit about E_∞ or strictly commutative ring spectra. Also a bit about E_n-spectra for $n < \infty$, especially $n = 1$ and $n = 3$.

I have slides for a talk which covers many of the above ideas. It could be adapted and extended to cover the rest of them.

2. K-theoretic invariants

- For a ring spectrum R, the algebraic K-theory $K(R)$.
- Also $THH(R)$ and its cyclotomic structure.
- Then $TR(R)$ and $TC(R)$, using the approach of Nikolaus and Scholze. These should come with maps $K(R) \to TC(R) \to TR(R) \to THH(R)$.
- Galois extensions of ring spectra, and interactions with the above invariants; descent theorems.
- Redshift calculations and conjectures of Ausoni and Rognes.
- The redshift theorem of Hahn and Wilson.

There is a survey paper by Akhil Mathew which might be a good source for some of this. It might be necessary to read the paper of Nikolaus and Scholze, but that is long, and I think that some parts can be done more easily now. We will certainly need to look at the main redshift paper of Hahn and Wilson.

3. Cyclotomic extensions

- Some ideas about ambidexterity.
- Higher cyclotomic extensions of ring spectra, constructed using ambidexterity.
- Cyclotomic completion of spectra.
- Interactions between K-theoretic invariants and cyclotomic extensions.
- Redshift in cyclotomic extensions.

There are two key papers by Carmeli, Schlank and Yanovski, joined by Ben-Moshe for the second paper.