Introduction to chromatic homotopy

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October 23, 2023

The Spanier-Whitehead category

- A *finite spectrum* is an expression $\Sigma^n X$, where X is a based finite simplicial complex, and $n \in \mathbb{Z}$. (This can be interpreted as a space if $n \geq 0$, but not necessarily if $n < 0$.) We write $\mathcal{F}$ for the class of finite spectra.
- We define $\mathcal{F}(\Sigma^n X, \Sigma^m Y) = \varinjlim_k [\Sigma^{n+k} X, \Sigma^{m+k} Y]$. This has a natural structure as a (finitely generated) Abelian group. There is a composition rule making $\mathcal{F}$ an additive category.
- This is an approximation to the homotopy category of finite complexes, and has a rich and interesting structure.
- Homology gives an isomorphism $\mathbb{Q} \otimes \mathcal{F}(X, Y) \to \text{Vect}_*(H_*(X; \mathbb{Q}), H_*(Y; \mathbb{Q}))$.
- The category $\mathcal{F}$ has formal properties similar to those of Vect_*: there are tensor products, duals and adjoints.
- It is very hard work to calculate $\mathcal{F}(X, Y)$, even in simple cases like $\mathcal{F}(S^d, S^0)$. This is known for $d \leq 100$ or so, but not for general d.
- There is also a category $\mathcal{S}$ of all spectra. Any spectrum is a filtered colimit of finite spectra.

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- In 1984, Ravenel made a set of conjectures about the category of spectra.
- With the exception of the Telescope Conjecture (TC), all the conjectures were proved by Devinatz, Hopkins and Smith.
  This led to a huge body of results in chromatic homotopy theory.
- It soon became the consensus that TC was probably false, and there was a programme by Mahowald, Ravenel and Schick to disprove it, but they could not complete the argument.
- A disproof was announced by Burklund, Hahn, Levy and Schlank in 2023.
- There are invariants $K(p,n)_*(X)$ of spectra X (for p prime and $n \geq 0$) called *Morava K-theory*. These play a central rôle in all the conjectures.
- Idea: focus on aspects of the category of spectra that are detected by $K(p,n)$ for a fixed (p,n).
- There are two subtly different versions of this: TC says they are the same.
- This is easy for $n = 0$, true for $n = 1$ and false for $n > 1$.
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Ordinary cohomology

- For any space X we have a cohomology ring $H^*(X)$
- For many spaces this can be described explicitly: for example, if $X = \{$ two-dimensional subspaces of $\mathbb{C}^4\}$ then $H^*(X) = \mathbb{Z}[c_1, c_2]/(c_1^3 - 2c_1c_2, c_1^2c_2 - c_2^2)$.
- We can also consider the scheme $X_H = \text{spec}(H^*(X))$, so $H^*(X)$ is the ring of functions on X_H.
- Now $f\colon X \to Y$ gives $f_H\colon X_H \to Y_H$ (depending only on the homotopy class) and $(X \amalg Y)_H = X_H \amalg Y_H$ and $(X \times Y)_H \sim X_H \times Y_H$.
- How good an invariant is this?
  - If $f_H\colon X_H \to Y_H$ is an isomorphism then f is a homotopy equivalence (subject to mild conditions).
  - The map $[X, Y] \to \text{Schemes}(X_H, Y_H) = \text{Rings}(H^*(Y), H^*(X))$ is typically far from being injective or surjective.
  - If $X_H \simeq Y_H$, that is only weak evidence for $X \simeq Y$.
- How to find better invariants?
  (a) Use Steenrod operations on $H^*(X; \mathbb{F}_p)$
  (b) Use generalised cohomology theories.
- But (a) is really part of (b).

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Formal groups — what are they good for?

- Every even periodic theory E gives a formal group P_E.
- The functor $E \mapsto P_E$ is not too far from being an equivalence.
- The most elementary examples of formal groups are the additive and multiplicative formal groups; these correspond to HP and KU. (Here $HP^i(X) = \prod_j H^{i+2j}(X)$.)
- Steenrod operations in $HP^0(X; \mathbb{F}_p)$ and Adams operations in $KU^0(X)$ are closely related to endomorphisms of the associated formal groups.
- The ring $MP^0(1)$ is naturally isomorphic to the Lazard ring, which plays a central role in formal group theory.
- The Morava K-theories $K(p, n)$ all have different formal groups.
- Together with $HP^0(X; \mathbb{F}_p)$ and $HP^0(X; \mathbb{Q})$ this gives all formal groups over fields up to Galois twisting.
- For many spaces X the scheme X_E can be described naturally in terms of P_E. For example, if $X = BU(n) = \{n - \text{dimensional subspaces of } \mathbb{C}^\infty\}$ then $X_E = (P_E^n)/\Sigma_n$.

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  - $G_e(R) = \{(u, v) \in \mathrm{Nil}(R)^2 \mid v - u^3 + uv^2 = 0\}$ (an elliptic curve)
- These are all functorial in R.
- We can define natural bijections $x_i \colon G_i(R) \to \mathrm{Nil}(R)$ by $x_a(a) = a$ and $x_m(u) = u - 1$ and $x_r(A) = s/c$ and $x_e(u, v) = u$.
- One can check that $x_i(a * b) = F_i(x_i(a), x_i(b))$ where $F_a(s, t) = s + t$ and $F_m(s, t) = s + t + st$ and $F_r(s, t) = (s + t)/(1 - st) = \sum_{i \geq 0} s^i t^i (s + t)$. (One cannot be so explicit for F_e.)
- The functors G_i are *formal groups*; the power series F_i are *formal group laws*.
- Axioms: $F(s, 0) = s,\ F(s, t) = F(t, s)$ and $F(F(s, t), u) = F(s, F(t, u))$.
- More general version: we have a ground ring k, and $G(R)$ is only functorial for k-algebras, and $F(s, t) \in k[\![s, t]\!]$.
- Example: for any $a \in k$ we have an FGL $F(s, t) = s + t + ast$ over k.

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Formal groups from even periodic theories

- $P = (\mathbb{C}[t] \setminus \{0\})/\mathbb{C}^\times = \{1-\dim\ \text{subspaces of}\ \mathbb{C}[t]\} = \mathbb{C}P^\infty$.
- This is a commutative topological monoid (with inverses up to homotopy).
- So P_E is a formal group scheme over $1_E = \text{spec}(E^0(1))$.
- We can calculate $E^*(\mathbb{C}P^n)$ by induction on n using Mayer-Vietoris. It follows that there exists x with $E^0(P) = E^0(1)[\![x]\!]$ (but there is no canonical choice of x).
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The Lazard ring

- Consider a formal power series $F(s,t) = \sum_{i,j} b_{ij} s^i t^j \in k[\![s,t]\!]$. When is this an FGL?
- For $F(s,0) = s$ we need $b_{i0} = \delta_{i,1}$. For $F(s,t) = F(t,s)$ we need $b_{ij} = b_{ji}$.
- Now $F(s,t) = s + t + b_{11}st + b_{12}(st^2 + s^2t) + b_{22}s^2t^2 + b_{13}(st^3 + s^3t) + O(5)$
- Using this we get $F(F(s,t),u) - F(s,F(t,u)) = (2b_{11}b_{12} + 3b_{13} - 2b_{22})(s-u)stu + O(5)$
- For an FGL we must have $2b_{11}b_{12} + 3b_{13} - 2b_{22}$. In terms of the parameters $a_1 = b_{11}$ and $a_2 = b_{12}$ and $a_3 = b_{22} - b_{13}$ we get $F(s,t) = s+t+a_1st+a_2st(s+t)+2(a_3-a_1a_2)st(s^2+st+t^2)+a_3s^2t^2+O(5)$.
- There are no more relations: any power series of the above form satisfies the FGL conditions up to errors of order 5.
- Lazard's theorem: we can continue to define $a_4, a_5, \dots$ so that $F(s,t)$ can be expressed in terms of the a_i, and no further relations are required to make the associativity axiom hold.
- Reformulation: over the Lazard ring $L = \mathbb{Z}[a_1, a_2, \dots]$ there is a universal formal group law F_u such that the resulting map $\text{Rings}(L,k) \to \text{FGL}(k)$ is bijective for all k.

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The chromatic filtration

- Fact: if $K(p,n)_*(X) = 0$, then $K(p,m)_*(X) = 0$ for all $m < n$ (including $K(p,0)_*(X) = H_*(X;\mathbb{Q})$).
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- However, there is an action of $\mathbb{Z}$ on $BP\langle n - 1\rangle$ and we can define $B(n - 1) = BP\langle n - 1\rangle^{h\mathbb{Z}}$. The new counterexample is $K(B(n - 1))$.
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The Telescope Conjecture again

- Fix (p,n) and put $\mathcal{C} = \{X \mid K(p,n)_*(X) = 0\}$.
- Say $X \in \mathcal{C}^f \subseteq \mathcal{C}$ if X is a filtered colimit of finite spectra in $\mathcal{C}$.
- Put $\widehat{\mathcal{L}} = \{Y \mid [\mathcal{C}, Y] = 0\} \simeq \mathcal{S}/\mathcal{C}$ and $\widehat{\mathcal{L}}^f = \{Y \mid [\mathcal{C}^f, Y] = 0\} \simeq \mathcal{S}/\mathcal{C}^f$. There are localisation functors $\widehat{L}\colon \mathcal{S} \to \widehat{\mathcal{L}}$ and $\widehat{L}^f\colon \mathcal{S} \to \widehat{\mathcal{L}}^f$.
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