§3: THH of cochains on the circle

Telescope conjecture reading group

So far...

- Dan introduced THH and TC, etc. for an $\mathbb{E}_1$-ring spectra R via the category $N\mathcal{A}ss^{\otimes}_{act}$ along with some other constructions, e.g. Tate, Frobenius, etc.
- TC of $\mathbb{E}_1$-rings featured in Proposition 1.1 and its proof reduces to checking a similar result for THH of $\mathbb{S}$-valued cochains.
- Note $\mathbb{S}^{BA}$ is an $\mathbb{E}_\infty$-ring spectrum. When R is $\mathbb{E}_\infty$ we have a simpler description of $\mathrm{THH}(R)$ as the pushout in $\mathrm{CAlg}(\mathrm{Sp})$ of R with $\mathbb{T}$:

  $$\mathrm{THH}(R) = R \otimes \mathbb{T}\,,$$

  alternatively:

  $$\mathrm{THH}(R) = R \otimes_{R \otimes R} R\,.$$

Proposition 1.1

Proposition 1.1

For any p-complete $\mathbb{E}_1$-ring R, the p-completion of $\mathrm{TC}(R)$ is in the thick subcategory generated by the p-completion of the fibre of the coassembly map:

$$\mathrm{TC}(R^{B\mathbb{Z}}) \longrightarrow \mathrm{TC}(R)^{B\mathbb{Z}}.$$

This will be useful in proving the following

Theorem B (Asymptotic constancy for $\mathrm{BP}\langle n\rangle$)

Fix a telescope $T(n+1)$ of a type $n+1$ p-local finite spectrum. Then for all $k \gg 0$ there is a commuting square:

$$\begin{array}{ccc} T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) & \longrightarrow & T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}} \\ \cong\downarrow & & \downarrow\cong \\ T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle^{B\mathbb{Z}}) & \longrightarrow & T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{B\mathbb{Z}} \end{array}.$$

Plan of attack

Preliminaries

§3.1: THH of cochains as a commutative algebra

§3.2: THH of cochains as a $\mathbb{T}$-equivariant commutative algebra

§3.3: THH of cochains as a cyclotomic spectrum

§3.4: Proof of Proposition 1.1

Plan of attack

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Assembly and coassembly

Coassembly

Let F be a contravariant homotopy functor, $F\colon \mathrm{Spaces}^{\mathrm{op}} \to \mathrm{Sp}$, then there is a zig-zag called the coassembly map:

$$\begin{aligned} F(X) &\to \lim_{(\Delta^p \to X) \in \Delta_{SX}^{\mathrm{op}}} F(\Delta^p) \\ &\xleftarrow{\sim} \lim_{(\Delta^p \to X) \in \Delta_{SX}^{\mathrm{op}}} F(*) \\ &\cong \mathrm{map}(|N\Delta_{SX}|_+, F(*)) \\ &\xleftarrow{\sim} \mathrm{map}(X_+, F(*)) \,. \end{aligned}$$

Examples

Take $F(-) := \mathrm{THH}(\mathbb{S}^{(-)})$ then the coassembly above gives a map:

$$\mathrm{THH}(\mathbb{S}^{BA}) \longrightarrow \mathrm{THH}(\mathbb{S})^{BA} \cong \mathbb{S}^{BA} \,.$$

Assembly and coassembly

Assembly

If F is a covariant homotopy functor, $F\colon \mathrm{Spaces} \to \mathrm{Sp}$, then there is similarly a zig-zag called the assembly map:

$$X_+ \wedge F(*) \longrightarrow F(X) \,.$$

Remarks

Assume $F(\emptyset) = *$. The coassembly (resp. assembly) map is characterised by the universal property that it is the universal approximation on the right (resp. left) by a linear functor, i.e. one that also preserves homotopy pushout squares.

Spherical Witt vectors

Recall

- The Witt vectors functor W takes an $\mathbb{F}_p$-algebra A and produces a characteristic 0 ring $W(A)$. E.g. $W(\mathbb{F}_p) = \mathbb{Z}_p$.
- An $\mathbb{F}_p$-algebra is *perfect* if the Frobenius $x \mapsto x^p$ is an automorphism. E.g. $\mathbb{F}_p$ is perfect.
- The spherical group ring functor $\mathbb{S}[-]$ produces a ring spectrum $\mathbb{S}[G]$ from a group G, which is commutative if G is.

Spherical Witt vectors adjunction

There is an adjunction between perfect $\mathbb{F}_p$-algebras and $\mathbb{E}_\infty$-ring spectra given by:

$$\mathbb{W}(-)\colon \mathrm{Perf} \rightleftarrows \mathrm{CAlg}(\mathrm{Sp})\colon \pi_0^\flat(-)\,.$$

□

Spherical Witt vectors cont.

The adjunction $\mathbb{W}(-) \dashv \pi_0^\flat(-)$

$$\mathbb{W}(-) \colon \mathrm{Perf} \rightleftarrows \mathrm{CAlg}(\mathrm{Sp}) \colon \pi_0^\flat(-)\,.$$

Definitions

- $\mathbb{W}(A) := \mathbb{S}[W(A)]$.
- $\pi_0^\flat(R) := \lim\left(\ldots \to \pi_0(R)/p \xrightarrow{F} \pi_0(R)/p \xrightarrow{F} \pi_0(R)/p\right)$.

Example

$\mathbb{W}(\mathbb{F}_p) = \mathbb{S}$.

Remarks

$\mathbb{W}$ is fully faithful and so Perf is a colocalisation of $\mathrm{CAlg}(\mathrm{Sp})$. The essential image of $\mathbb{W}$ consists of those $R \in \mathrm{CAlg}(\mathrm{Sp})$ such that $\mathbb{F}_p \otimes R$ is a discrete perfect $\mathbb{F}_p$-algebra. We then have $R \cong \mathbb{W}(R \otimes \mathbb{F}_p)$.

Plan of attack

Preliminaries

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§3.4: Proof of Proposition 1.1

Lattice of $\mathbb{Z}_p$-modules

- First step now is to study the commutative algebras $\mathrm{THH}(\mathbb{S}^{Bp^k\mathbb{Z}})$.
- Will in fact study $\mathrm{THH}(\mathbb{S}^{BM})$ where M is any discrete finite projective $\mathbb{Z}_p$-module.
- This is justified since for a free finite rank $\mathbb{Z}$-module M the map $BM \to BM_p$ is an equivalence on p-complete suspension spectra since it is one on $\mathbb{F}_p$-homology. Hence so too is $\mathbb{S}^{BM_p} \to \mathbb{S}^{BM}$.

Definition

Write $\mathrm{Latt}_{\mathbb{Z}_p}$ for the category of discrete finite projective $\mathbb{Z}_p$-modules.

Identifying Witt vectors of continuous functions

- $C^0(A)$ denotes $\mathbb{F}_p$-valued continuous functions on the $\mathbb{Z}_p$-module A.
- A^δ is A equipped with the discrete topology.
- Ω_e is based loops.

Lemma 3.5

There's a commutative diagram of commutative algebras:

$$\begin{array}{ccc} \mathbb{W}C^0(A) & \xrightarrow{i} & \mathbb{W}C^0(A^\delta) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S} & \longrightarrow & \mathbb{S}^{\Omega_e BA} \end{array}$$

natural in $A \in \mathrm{Latt}_{\mathbb{Z}_p}$.

Proof of Lemma 3.5

$$
\begin{array}{ccccc}
\operatorname{colim}_k \mathbb{F}_p \otimes_{\mathbb{F}_p^{BA/p^k}} \mathbb{F}_p & \xrightarrow[(1)]{\cong} & \mathbb{F}_p \otimes_{\mathbb{F}_p^{BA}} \mathbb{F}_p & \xleftarrow[(2)]{\cong} & (\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}) \otimes \mathbb{F}_p \\
\cong \downarrow (3) & & \downarrow & & \downarrow \\
\operatorname{colim}_k \mathbb{F}_p^{\Omega_e BA/p^k} & \longrightarrow & \mathbb{F}_p^{\Omega_e BA} & \xleftarrow[(4)]{\cong} & \mathbb{S}^{\Omega_e BA} \otimes \mathbb{F}_p \\
\cong \downarrow (5) & & \cong \downarrow (6) & & \\
C^0(A) & \xrightarrow{i} & C^0(A^\delta) & &
\end{array}
$$

- (1) is an iso since $H_c^*(A; \mathbb{F}_p) \cong H^*(A; \mathbb{F}_p)$.
- (2) and (4) are isos since $\mathbb{F}_p \otimes -$ commutes with finite limits and arbitray products that are uniformly bounded below.
- (3) is an iso by convergence of the Eilenberg-Moore spectral sequence.
- (5) and (6) are by definition.

Proof of Lemma 3.5 cont.

So we have a square:

$$\begin{array}{ccc} (\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}) \otimes \mathbb{F}_p & \longrightarrow & \mathbb{S}^{\Omega_e BA} \otimes \mathbb{F}_p \\ \cong \downarrow & & \downarrow \cong \\ C^0(A) & \xrightarrow{\quad i \quad} & C^0(A^\delta) \end{array}$$

and recall the essential image of $\mathbb{W}$ were those $R \in \mathrm{CAlg}(\mathrm{Sp})$ such that $\mathbb{F}_p \otimes R$ is a discrete $\mathbb{F}_p$-algebra. And so we get:

$$\begin{array}{ccc} \mathbb{W}C^0(A) & \xrightarrow{i} & \mathbb{W}C^0(A^\delta) \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S} & \longrightarrow & \mathbb{S}^{\Omega_e BA} \end{array}$$

$\mathrm{THH}(\mathbb{S}^{BA})$ as an $\mathbb{S}^{BA}$-algebra

Lemma 3.6

There is a commutative diagram of commutative $\mathbb{S}^{BA}$-algebras:

$$\begin{array}{ccc} \mathbb{S}^{BA} \otimes \mathbb{W}C^0(A) & \longrightarrow & \mathbb{S}^{BA} \otimes \mathbb{W}C^0(A^\delta) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{THH}(\mathbb{S}^{BA}) & \longrightarrow & \mathbb{S}^{\mathcal{L}BA} \end{array}$$

natural in $A \in \mathrm{Latt}_{\mathbb{Z}_p}$.

Proof of Lemma 3.6

$$
\begin{array}{ccccc}
\mathbb{S}^{BA} \otimes \mathbb{W}C^0(A) & \longrightarrow & \mathbb{S}^{BA} \otimes \mathbb{W}C^0(A^\delta) & & \\
\cong \downarrow & & \downarrow \cong & & \\
\mathbb{S}^{BA} \otimes (\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}) & \longrightarrow & \mathbb{S}^{BA} \otimes \mathbb{S}^{\Omega_e BA} & \xrightarrow{\cong} & \mathbb{S}^{BA \times \Omega_e BA} \\
\cong \downarrow & & & & \downarrow \cong \\
\mathrm{THH}(\mathbb{S}^{BA}) & & \xrightarrow{\quad \mathrm{Id} \quad} & & \mathbb{S}^{\mathcal{L}BA}
\end{array}
$$

- The bottom left horizontal map is an assembly map for the $\mathbb{T}$-shaped colimit THH along $\mathbb{S}^{(-)}$.
- The rightmost vertical map is an iso by the map $\mathcal{L}G \cong G \times \Omega_e G$ for G grouplike.
- The top square is $\mathbb{S}^{BA}$ tensored with Lemma 3.5.
- The bottom left vertical map is an iso because ?
- ??? They also claim to use $\pi_0^\flat(\mathbb{S}^{BA} \otimes R) \cong \pi_0^\flat(R)$ somewhere.

$\mathrm{THH}(\mathbb{S}^{BA})$ restricted to 0

Lemma 3.7

There is a pushout square of commutative algebras:

$$\begin{array}{ccc} \mathbb{W}(C^0(A)) & \xrightarrow{\mathbb{W}(ev_0)} & \mathbb{W}(\mathbb{F}_p) \\ \downarrow & & \downarrow \\ \mathrm{THH}(\mathbb{S}^{BA}) & \longrightarrow & \mathbb{S}^{BA} \end{array}$$

natural in $A \in \mathrm{Latt}_{\mathbb{Z}_p}$.

Proof of Lemma 3.7

- $\pi_0^\flat(\mathbb{S}^{BA}) = \mathbb{F}_p$ since $\pi_0^\flat$ is a right adjoint with values in a 1-category.
- Result follows from Lemma 3.6 and the fact that $\mathbb{S}^{BA} \to \mathrm{THH}(\mathbb{S}^{BA}) \to \mathbb{S}^{BA}$ is the identity.

Plan of attack

Preliminaries

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Some more preliminaries

Definition

Let $w \in \mathbb{Z}_p$ and $B\mathbb{Z}_p(w)$ denote $B\mathbb{Z}_p$ with $\mathbb{T} = B\mathbb{Z}$ action left via multiplication from $B\mathbb{Z} \to B\mathbb{Z}_p$ induced by $1 \mapsto w$.

Examples

There is a $\mathbb{T}$-equivariant decomposition of the free loop space on the p-adic circle:

$$\mathcal{L}B\mathbb{Z}_p \cong \coprod_{w \in \mathbb{Z}_p} B\mathbb{Z}_p(w)\,.$$

On the connected component corresponding to $w \colon B\mathbb{Z} \to B\mathbb{Z}_p$ the $\mathbb{T}$ action is the 'w-speed' rotation.

Some more preliminaries

We let:

- $\eta \in \pi_1(\mathbb{S})$ be the Hopf element.
- $\epsilon \in \pi_{-1}\mathbb{S}^{B\mathbb{Z}}$ be the class corresponding to $1 \in H^1(B\mathbb{Z}; \mathbb{Z})$.
- $\zeta \in \pi_{-1}\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ be the image of ϵ under $\mathbb{S}^{B\mathbb{Z}} \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$.
- σ be the Connes operator: given a $\mathbb{T}$-equivariant spectrum X there is a degree 1 self map:

$$\sigma\colon \Sigma X \to X$$

obtained by viewing X as a $\mathbb{S}[\mathbb{T}]$ module then σ corresponds to the pointed identity map $S^1 \to \mathbb{T}$ in $\pi_1\mathbb{S}[\mathbb{T}]$. This then induces $\sigma\colon \pi_n X \to \pi_{n+1} X$.

Lemma 3.11

Lemma 3.11

In $\pi_0 \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ we have $\sigma(\zeta) = (1 + \eta\zeta) \cdot \mathrm{Id}_{\mathbb{Z}_p}$.

Proof of Lemma 3.11

- From Lemma 3.6 the assembly map $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p}) \to \mathbb{S}^{\mathcal{L}B\mathbb{Z}_p}$ is injective so can compute $\sigma(\zeta)$ in the target instead.
- Using $\mathcal{L}B\mathbb{Z}_p \cong \coprod_{w \in \mathbb{Z}_p} B\mathbb{Z}_p(w)$ and the fact $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ is dense we reduce to computing $\sigma(\epsilon)$ in $\pi_0 \mathbb{S}^{B\mathbb{Z}(w)}$ where $w \in \mathbb{Z}$.
- Since $\mathbb{S}^{B\mathbb{Z}(w)} = w^* \mathbb{S}^{\mathbb{T}}$ for the degree w map $\mathbb{T} \to \mathbb{T}$ sends $\sigma \mapsto w\sigma$ we can instead show that:

$$\sigma(\epsilon) = 1 + \eta\epsilon \in \pi_* \mathbb{S}^{\mathbb{T}}.$$

- They then claim after rationalization this is straightforward and the general case follows from $\sigma \circ \sigma = \eta\sigma$.

p-speed action on THH equivalent to base changed THH

Construction 3.12

For $R \in \mathrm{CAlg}(\mathrm{Sp})$ tensoring by the p-fold cover $p\colon \mathbb{T} \to \mathbb{T}/C_p \cong \mathbb{T}$ yields a map of $\mathbb{T}$-equivariant commutative algebras:

$$\psi_p\colon \mathrm{THH}(R) \to p^*\mathrm{THH}(R)$$

where $p^*\colon \mathrm{Sp}^{B\mathbb{T}/C_p} \to \mathrm{Sp}^{B\mathbb{T}}$.

Lemma 3.13

The map ψ_p refines to a $\mathbb{T}$-equivariant map:

$$\psi_p\colon \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})_{|p\mathbb{Z}_p} \xrightarrow{\cong} p^*\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$$

inducing $\mathrm{res}_p\colon C^0(p\mathbb{Z}_p) \to C^0(\mathbb{Z}_p)$ on $\pi_0^\flat$. In particular the C_p action on $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})_{|p\mathbb{Z}_p}$ is trivialisable.

Proof of Lemma 3.13

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The map ψ_p refines to a $\mathbb{T}$-equivariant map:

$$\psi_p \colon \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})_{|p\mathbb{Z}_p} \xrightarrow{\cong} p^* \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$$

inducing $\mathrm{res}_p \colon C^0(p\mathbb{Z}_p) \to C^0(\mathbb{Z}_p)$ on $\pi_0^\flat$. In particular the C_p action on $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})_{|p\mathbb{Z}_p}$ is trivialisable.

Proof of Lemma 3.13

- The assembly map $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathbb{S}^{\mathcal{L}B\mathbb{Z}}$ are injective on homotopy we can prove the claim in $\mathbb{S}^{\mathcal{L}B\mathbb{Z}}$.
- Precomposition by the degree p map $S^1 \to S^1$ sends the circle at component $a \in \mathbb{Z}_p$ to the circle at component $pa \in \mathbb{Z}_p$ isomorphically. □

$\mathrm{THH}(\mathbb{S}^{BA})$ restricted to $p^k\mathbb{Z}_p^\times$

Lemma 3.14

For each $k \geq 0$ there is an iso of $\mathbb{W}C^0(p^k\mathbb{Z}_p^\times)$-modules in $\mathrm{Sp}^{B\mathbb{T}}$:

$$\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})_{|p^k\mathbb{Z}_p^\times} \cong \mathbb{W}C^0(p^k\mathbb{Z}_p^\times) \otimes \Sigma^{-1}\mathbb{S}[\mathbb{T}/C_{p^k}]\,.$$

Proof of Lemma 3.14

- By Lemma 3.13 we can reduce to $k = 0$.
- Restricting $\zeta \in \pi_1\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})$ down to a class in $\pi_1\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})_{|\mathbb{Z}_p^\times}$ we can construct a $\mathbb{T}$-equivariant map of $\mathbb{W}C^0(\mathbb{Z}_p^\times)$-modules:

$$z := \mathbb{W}C^0(\mathbb{Z}_p^\times) \otimes \Sigma^{-1}\mathbb{S}[\mathbb{T}] \longrightarrow \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})_{|\mathbb{Z}_p^\times}\,.$$

- On homotopy this is:

$$z\colon\ \left(\pi_*\mathbb{W}C^0(\mathbb{Z}_p^\times)\right)\{[*],[\mathbb{T}]\} \longrightarrow C^0(\mathbb{Z}_p^\times)\{1,\zeta\}\,.$$

Proof of Lemma 3.14 cont.

- On homotopy this is:

$$z\colon \left(\pi_* \mathbb{W}C^0(\mathbb{Z}_p^\times)\right)\{[*],[\mathbb{T}]\} \longrightarrow C^0(\mathbb{Z}_p^\times)\{1,\zeta\}$$

  with $z([*]) = \zeta$.
- To compute $z([\mathbb{T}])$ we have by Lemma 3.11:

$$z([\mathbb{T}]) = z(\sigma([*])) = \sigma(z([*])) = \sigma(\zeta) = (1+\eta\zeta)\cdot \mathrm{Id}_{\mathbb{Z}_p}$$

- The $\mathbb{W}C^0(p^k\mathbb{Z}_p^\times)$-module iso follows since $\mathrm{Id}_{\mathbb{Z}_p}$ is a unit when restricted to $\mathbb{Z}_p^\times$.

Lemma 3.15

Let $R \in \mathrm{CAlg}(\mathrm{Sp})^{BC_p}$ be bounded below. In the span diagram:

$$\pi_0^\flat R \longleftarrow \pi_0^\flat R^{hC_p} \longrightarrow \pi_0^\flat R^{tC_p}$$

the left arrow is an iso if the C_p-action on $\pi_0 R$ is trivial, e.g. when the C_p-action extends to a $\mathbb{T}$ action. The right arrow is an iso if the C_p-action is trivial.

Proof of Lemma 3.15

- The left arrow being iso, under the assumption, follows since $\pi_0^\flat$ is a right adjoint factoring through π_0 and taking values in a 1-category.

Proof of Lemma 3.15 cont.

$$\pi_0^\flat R^{hC_p} \longrightarrow \pi_0^\flat R^{tC_p}$$

- The Postnikov tower refines the map $\mathbb{S} \to \mathbb{Z}_p$ to a tower of square-free extensions.
- $(-)^{hC_p}$ and $(-)^{tC_p}$ are exact and commute with uniformly bounded below limits.
- $\pi_0^\flat$ sends square-zero extensions to isos.
- These reduce us to check for R replaced with $\mathbb{Z}_p \otimes R$.
- The map is then just

$$\pi_*(F_p \otimes R)[\![t]\!] \to \pi_*(\mathbb{F}_p \otimes R)(\!(t)\!)$$

  with $|t| = -2$.
- Since R is bounded below, $|t| < 0$ this is a nil-extension in degree 0 so we're done.

Plan of attack

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A diagram and the Cyclotomic Frobenius

We can patch together Lemmas 3.13, 3.14, and 3.15 to get a diagram:

$$\begin{array}{ccccc}
\pi_0^\flat \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) & \longleftarrow & \pi_0^\flat \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})^{hC_p} & \longrightarrow & \pi_0 \mathrm{THH}(\mathbb{S}^{b\mathbb{Z}})^{tC_p} \\
\cong \downarrow & & \downarrow \cong & & \downarrow \cong \\
C^0(\mathbb{Z}_p) & \xleftarrow{\mathrm{Id}} & C^0(\mathbb{Z}_p) & \xrightarrow{(-)_{|p\mathbb{Z}_p}} & C^0(p\mathbb{Z}_p)
\end{array}$$

Cyclotomic Frobenius

Recall for $R \in \mathrm{CAlg}(\mathrm{Sp})$ there's a unique $\mathbb{T}$-equivariant map of commutative algebras:

$$\varphi \colon \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p} .$$

The cyclotomic Frobenius is an iso

Proposition 3.18

The cyclotomic Frobenius $\varphi\colon \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p}) \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})^{tC_p}$ is an iso and $\pi_0^\flat(\varphi) = \mathrm{res}_{1/p}$ where $1/p\colon p\mathbb{Z}_p \to \mathbb{Z}_p$.

Proof of Proposition 3.18

$$\begin{array}{ccccccc}
\mathbb{S}^{B\mathbb{Z}_p} & \xrightarrow{\Delta_p} & ((\mathbb{S}^{B\mathbb{Z}_p})^{\otimes p})^{tC_p} & \xrightarrow{m} & (\mathbb{S}^{B\mathbb{Z}_p})^{tC_p} & \xleftarrow[\text{can}]{\cong} & \mathbb{S}^{B\mathbb{Z}_p} \\
\downarrow & & \downarrow (\mu_p^*)^{tC_p} & & \downarrow & & \downarrow \\
\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p}) & \xrightarrow{\varphi} & \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})^{tC_p} & \xrightarrow{\psi_p^{tC_p}} & (p^*\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p}))^{tC_p} & \xleftarrow{\cong} & \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})
\end{array}$$

- Left square is definition of the cyclotomic Frobenius. Middle square is tensoring $\mathbb{S}^{B\mathbb{Z}_p}$ with the ses $C_p \to \mathbb{T} \to \mathbb{T}/C_p$ and applying $(-)^{tC_p}$. Right square is constructed using a trivialization of the C_p action on $* \to \mathbb{T}/C_p$.

Proof of Proposition 3.18 cont.

$$
\begin{array}{ccccccc}
\mathbb{S}^{B\mathbb{Z}_p} & \xrightarrow{\Delta_p} & ((\mathbb{S}^{B\mathbb{Z}_p})^{\otimes p})^{tC_p} & \xrightarrow{m} & (\mathbb{S}^{B\mathbb{Z}_p})^{tC_p} & \xleftarrow[\text{can}]{\cong} & \mathbb{S}^{B\mathbb{Z}_p} \\
\downarrow & & \downarrow{\scriptstyle (\mu_p^*)^{tC_p}} & & \downarrow & & \downarrow \\
\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p}) & \xrightarrow{\varphi} & \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})^{tC_p} & \xrightarrow{\psi_p^{tC_p}} & (p^*\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p}))^{tC_p} & \xleftarrow{\cong} & \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})
\end{array}
$$

- Allen Yuan showed the indicated isos. By Lemmas 3.13 and 3.14 $\psi_p^{tC_p}$ is an iso.
- The composite $m \circ \Delta_p$ is the Tate valued Frobenius.
- The composite $\mathrm{can}^{-1} \circ m \circ \Delta_p$ is the identity on $\mathbb{S}$.
- Naturality of the limit over $B\mathbb{Z}_p$ now implies that $\mathrm{can}^{-1} \circ m \circ \Delta_p$ is the identity on $\mathbb{S}^{B\mathbb{Z}_p}$ too.
- The universal property of THH: $R \to \mathrm{THH}(R)$ being initial amongst $\mathbb{E}_\infty$-maps from R to $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-rings shows φ is iso.

Definition of TR as a corepresentable

Lemma/Definition of TR

Let $L_{\langle p^\infty \rangle}\mathbb{S}$ be the cyclotomic spectrum with underlying $\mathbb{T}$-equivariant spectrum given by:

$$\bigoplus_{j \geq 0} \mathbb{S}[\mathbb{T}/C_{p^j}] \,.$$

The Frobenius $\varphi\colon L_{\langle p^\infty \rangle}\mathbb{S} \longrightarrow L_{\langle p^\infty \rangle}\mathbb{S}^{tC_p}$ is the iso (by the Segal conjecture) given by the sum of the composites:

$$\mathbb{S}[\mathbb{T}/C_{p^j}] \to (\mathbb{S}[\mathbb{T}/C_{p^{j+1}}])^{hC_p} \to (\mathbb{S}[\mathbb{T}/C_{p^{j+1}}])^{tC_p} \,.$$

$L_{\langle p^\infty \rangle}\mathbb{S}$ then corepresents $TR(-)$ in CycSp_+.

Proposition 3.19

There is a fibre sequence of cyclotomic spectra:

$$\mathbb{W}C^0(\mathbb{Z}_p^\times) \otimes \Sigma^{-1} L_{\langle p^\infty \rangle}\mathbb{S} \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}$$

Proof of Proposition 3.19

Write F for the fibre:

$$F \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}.$$

- F is the fibre of commutative algebras in CycSp so it is a non-unital algebra in CycSp.
- By Lemma 3.7 F is iso to $\bigoplus_{k\geq 0} F_{|p^k\mathbb{Z}_p^\times}$ as $\mathbb{T}$-equivariant non-unital commutative algebras.
- By Lemma 3.14 we identified an iso of $\mathbb{W}C^0(\mathbb{Z}_p^\times)$-modules:

$$F_{|p^k\mathbb{Z}_p} \cong \mathbb{W}C^0(\mathbb{Z}_p^\times) \otimes \Sigma^{-1}\mathbb{S}[\mathbb{T}/C_{p^k}].$$

- By Proposition 3.18 the cyclotomic Frobenius on F breaks up as a sum of isos $F_{|p^k\mathbb{Z}_p^\times} \cong (F_{|p^{k+1}\mathbb{Z}_p^\times})^{tC_p}$.
- The splitting of the Frobenius gives the result (by Lemma 2.7 we haven't seent).

Plan of attack

Main result

Corollary 3.21

Let $R \in \mathrm{Alg}(\mathrm{Sp})$ be connective. Then $\mathbb{W}C^0(\mathbb{Z}_p^\times) \otimes TC(R)$ is in the thick subcategory generated by the fibre of the coassembly map:

$$TC(R^{B\mathbb{Z}}) \longrightarrow TC(R)^{B\mathbb{Z}}.$$

Proof of Corollary 3.21

- There's a fibre sequence:

$$\mathrm{TC} \longrightarrow \mathrm{TR} \xrightarrow{1-F} \mathrm{TR}$$

(also another way to define TC) from which we get a fibre sequence on corepresenting objects:

$$L_{\langle p^\infty \rangle}\mathbb{S} \longrightarrow L_{\langle p^\infty \rangle}\mathbb{S} \longrightarrow \mathbb{S}.$$

Proof of Corollary 3.21

- Tensor this with $\mathbb{W}C^0(\mathbb{Z}_p^\times)$ to get:

$$\mathbb{W}C^0(\mathbb{Z}_p^\times) \otimes L_{\langle p^\infty \rangle}\mathbb{S} \longrightarrow \mathbb{W}C^0(\mathbb{Z}_p^\times) \otimes L_{\langle p^\infty \rangle}\mathbb{S} \longrightarrow \mathbb{W}C^0(\mathbb{Z}_p^\times)\,.$$

- The first two terms are the (suspension) of the fibre of the $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ coassembly map, hence $\mathbb{W}C^0(\mathbb{Z}_p^\times)$ is in the thick subcategory generated by the fibre of the $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ coassembly map.

Proof of Corollary 3.21

- Tensoring by $\mathrm{THH}(R)$ the fibre sequence $F \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}$ and use THH is monoidal along with $\mathbb{S}^{B\mathbb{Z}} \otimes R \cong R^{B\mathbb{Z}}$ to get the following cofibre sequence:

$$F \otimes \mathrm{THH}(R) \to \mathrm{THH}(R^{B\mathbb{Z}}) \to \mathrm{THH}(R)^{B\mathbb{Z}}$$

- Since TC preserves colimits and $\mathbb{W}C^0(\mathbb{Z}_p)$ is a p-adic sum of spheres after applying TC we get the sequence:

$$F \otimes TC(R) \to \mathrm{TC}(R^{B\mathbb{Z}}) \to \mathrm{TC}(R)^{B\mathbb{Z}}$$

from which the result follows.